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A Liaison among Inclusion-Exclusion, Probability-Ready Expressions and Boole-Shannon Expansion for Multi-State Reliability

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Abstract. This paper deals with an emergent variant of the classical problem of computing the probability of the union of n events, or equivalently the expectation of the disjunction (ORing) of n indicator variables for these events, *i.e*., the probability of this disjunction being equal to one. The variant considered herein deals with multi-valued variables, in which the required probability stands for the reliability of a multi-state delivery network (MSDN), whose system success is a two-valued function expressed in terms of multi-valued component successes. The paper discusses four approaches for handling the afore-mentioned problem in terms of a standard MSDN, whose success is known in minimal form as the disjunction of its prime implicants, which are the minimal paths of the pertinent network. The paper briefly outlines and discusses two standard solutions via the utilization of the multi-state inclusion-exclusion (MS-IE) principle, and via the construction of a multi-state probability-ready expression (MS-PRE). We successfully extrapolate the PRE concept from the two-valued logical domain to the multi-valued logical domain, and employ it for a direct transformation of a random logical expression, on a one-to-one basis, to its statistical expectation form, simply by replacing all logic variables by their statistical expectations, and also substituting arithmetic multiplication and addition for their logical counterparts (ANDing and ORing). The main contribution of the paper is to provide two systematic and more efficient procedures for handling the required problem. The first procedure uses the multi-state Boole-Shannon expansion, while the second procedure applies the MS-IE principle to fewer (factored or composite) paths that are set (at minimal cost) to PRE form. The four approaches discussed are illustrated with a detailed symbolic example of a real-case study, and each of them produces a more precise version of the same numerical value that was obtained earlier by the method of recursive sum of disjoint products (RSDP). The paper is a part of an on-going activity that strives to provide a pedagogical treatment of multi-state reliability problems, and to establish a clear and insightful interrelationship between the two-state modeling and the multi-state one by stressing that multi-valued concepts are natural and simple extensions of two-valued ones.

Keywords: Network reliability, Inclusion-exclusion, Probability-ready expression, Boole-Shannon expansion, Multi-state system, Multiple-valued logic, Symbolic expression, Multi-State Delivery Network.

1. Introduction

This paper deals with a fundamental problem of multi-state reliability, which pertains to the computation of the expectation of the logical expression of a multi-state disjunctive normal form (DNF). Currently, the most computationally efficient method for handling this problem is an automated implementation of the method of the recursive sum of disjoint products (RSDP) [1–3]. We present a tutorial discussion of four approaches (in descending order of complexity) for solving this problem. These approaches are based on (a) the multi-state inclusion-exclusion (MS-IE) principle, (b) the concept of a multi-state probability-ready expression (MS-PRE), (c) the multi-state Boole-Shannon (MS-BS) expansion, and (d) a novel approach that combines factoring, MS-IE and MS-PRE. Our third approach (occasionally associated with the second) captures the essence of the RSDP method.

This paper is a part of an on-going activity that strives to provide a pedagogical treatment of multi-state reliability problems. We aspire to establish a clear and insightful interrelationship between the two-state modeling and the multi-state one by stressing that multi-valued concepts are natural and simple extensions of two-valued ones. Moreover, we hope to extend the concept of the sum of disjoint products (SDP) in the multi-state domain to the more encompassing one of a probability-ready expression (PRE). Finally, we need to provide a useful liaison among MS-IE, MS-PRE, and MS-BS.

The organization of the remainder of this paper is as follows. Section 2 presents important pertinent assumptions and notation. It further introduces a running example of a multi-state delivery network (MSDN) with multiple suppliers, borrowed from Lin *et al*. [2]. Section 3 introduces the multi-state inclusion-exclusion (MS-IE) principle, while Section 4 extends the concept of a multi-state probability-ready expression (MS-PRE) from the binary to the multi-state case. The two sections outline the application of their pertinent methods to the running example. Section 5 presents the multi-state Boole-Shannon, and demonstrate it in terms of the running example. Section 7 applies the multi-state inclusion-exclusion (MS-IE) principle to the same example using fewer (factored or composite) paths that are set (at minimal cost) to PRE form. Section 7 discusses the results obtained, while Section 8 concludes the paper.

2. Assumptions, Notation and Specification of a Running Example

2.1 Assumptions

- The model considered is one of a system with binary output and multistate components, specified by the structure or success function $S(\boldsymbol{X})$ [4]

$$S: \{0,1,\cdots,m_1\} \times \{0,1,\cdots,m_2\} \times \ldots \times \{0,1,\cdots,m_n\} \rightarrow \{0,1\}. \qquad (1)$$

- The system is generally non-homogeneous, *i.e.*, the number of system states (two) and the numbers of component states $(m_1+1), (m_2+1), \cdots, (m_n+1)$ might differ. When these numbers have a common value, the system reduces to a homogeneous one.

- The system is a non-repairable one with statistically independent non-identical (heterogeneous) components.

The system is a coherent one enjoying the properties of causality, monotonicity, and component relevancy [4-9].

2.2 Notation

Symbol	Description
X_k	A multivalued input variable representing component k $(1 \le k \le n)$, where $X_k \in \{0,1,\ldots,m_k\}$, and $m_k \ge 1$ is the highest value of X_k.
$X_k\{j\}$	A binary variable representing instant j of X_k $$X_k\{j\} = \{X_k = j\},$$

	i.e., $X_k\{j\} = 1$ if $X_k = j$ and $X_k\{j\} = 0$ if $X_k \neq j$. The instances $X_k\{j\}$ for $\{ 0 \leq j \leq m_k\}$ form an orthonormal set, namely, for $\{1 \leq k \leq n\}$ $\bigvee_{j=0}^{m_k} X_k\{j\} = 1, \quad (2a)$ $X_k(j_1)\, X_k(j_2) = 0 \ \text{ for } \ j_1 \neq j_2 \ . \quad (2b)$ Orthonormality is very useful in constructing inverses or complements. The complement of the union of certain instances is the union of the complementary instances. In particular, the complement of $X_k\{\geq j\} = X_k\{j, j+1, \ldots, m_k\}$ is $X_k\{< j\} = X_k\{0, 1, \ldots, j-1\}$.
$X_k\{\geq j\}$	An upper value of X_k $\{0 \leq j \leq m_k\}$: $X_k\{\geq j\} = X_k\{j, j+1, \ldots, m_k\} = \bigvee_{i=j}^{m_k} X_k\{i\} = X_k\{j\} \vee X_k\{j+1\} \vee \ldots \vee X_k\{m_k\}. \quad (3)$ The value $X_k\{\geq 0\}$ is identically 1. The set $X_k\{\geq j\}$ for $\{1 \leq j \leq m_k\}$ is neither independent nor disjoint, and hence it is difficult to be handled mathematically, but it is very convenient for translating the verbal or map/tabular description of a coherent component into a mathematical form when viewing component success at level j. The complement of $X_k\{\geq j\}$ is $X_k\{< j\} = X\{0, 1, \ldots, j-1\} = X_k\{0\} \vee X_k\{1\} \ldots \vee X_k\{j-1\} = X_k\{k \leq (j-1)\}. \quad (4)$
$X_k\{\leq j\}$	A lower value of X_k $\{0 \leq j \leq m_k\}$: $X_k\{\leq j\} = X_k\{0, 1, \ldots, j\} = \bigvee_{i=0}^{j} X_k\{i\} = X_k\{0\} \vee X_k\{1\} \ldots \vee X_k\{j-1\} \vee X_k\{j\}. \quad (5)$ The value $X_k\{\leq m_k\}$ is identically 1. The set $X_k\{\leq j\}$ for $\{0 \leq j \leq (m_k - 1)\}$ is neither independent nor disjoint, and hence it is not convenient for mathematical manipulation though it is suitable for expressing component failure at level $(j+1)$. Instances, upper values and lower values are related by $X_k\{j\} = X_k\{\geq j\}\, X_k\{< (j+1)\} = X_k\{\geq j\}\, \bar{X}_k\{\geq (j+1)\} = X_k\{\leq j\}\, X_k\{> (j-1)\}$ $= X_k\{\leq j\}\, \bar{X}_k\{\leq (j-1)\}. \quad (6)$
S	A binary output variable representing the system, where $S \in \{0, 1\}$. The function $S(\boldsymbol{X})$ is usually called the system success or the structure function. Its complement $\bar{S}(\boldsymbol{X})$ is called system failure, and is also a binary variable. The logical sum and arithmetic sum of success and failure are both equal to 1, namely $(S(\boldsymbol{X}) \vee \bar{S}(\boldsymbol{X})) = (S(\boldsymbol{X}) + \bar{S}(\boldsymbol{X})) = 1. \quad (7)$

2.3 Specifications for a Running Example

Lin *et al.* [2] studied a specific multi-state delivery network (MSDN) with multiple suppliers, one market, multiple transfer centers and eight branches. They derived an expression of system success for specific data of delivery costs, probability distributions of all branches, available capacities, suppliers' production capacities, deterioration rate vector for the minimal paths obtained, demand, and budget. They presented the final multi-state success in their Table 2, which is expressed below, with an appropriate translation of notation

$$S = X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_3\{\geq 3\}\, X_7\{\geq 3\} \vee X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{\geq 2\} \vee X_2\{\geq 3\}\, X_7\{\geq 3\} \vee X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2 \vee X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_4\{\geq 2\} X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 2\} X_6\{\geq 2\}\, X_7\{\geq 3\}. \quad (8)$$

Note that the expression of system success S in (8) reveals clearly that it pertains to a coherent system. The expression comprises eight distinct prime implicants, none of which subsumes (can be absorbed) in another. Each prime implicant is a product of solely upper values $X_k\{\geq j\}$ of various variables. For convenience, we rearrange the terms in (8), to let products with fewer variable instances appear first

$$S = X_3\{\geq 3\}\, X_7\, \{\geq 3\} \,\vee\, X_2\{\geq 3\}\, X_7\{\geq 3\}$$
$$\vee\, X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}$$
$$\vee\, X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}$$
$$\vee\, X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{\geq 2\}$$
$$\vee\, X_1\{\geq 2\} X_2\{\geq 2\} X_3\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 3\}$$
$$\vee\, X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}$$
$$\vee\, X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_4\{\geq 2\} X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\} \tag{8a}$$

Table 1. Numerical values for the expectations of various variable instances, computed from data given in Lin *et al.* [2].

$X_1\{\geq 2\}$	0.897	$X_3\{\geq 3\}$	0.905
$X_2\{\geq 3\}$	0.892	$X_3\{\geq 2\}$	0.953
$X_2\{< 3\}$	0.108	$X_3\{2\}$	0.048
$X_2\{\geq 2\}$	0.965	$X_3\{< 3\}$	0.095
$X_2\{2\}$	0.073	$X_4\{\geq 2\}$	0.863
$X_4\{< 2\}$	0.137	$X_5\{\geq 3\}$	0.903
$X_5\{< 3\}$	0.097	$X_6\{\geq 2\}$	0.943
$X_7\{\geq 2\}$	0.945	$X_7\{\geq 3\}$	0.884
$X_7\{2\}$	0.061	$X_7\{< 3\}$	0.116
$X_8\{\geq 3\}$	0.906	$X_8\{\geq 2\}$	0.965
$X_8\{2\}$	0.059	$X_8\{< 3\}$	0.094

The numerical values for the expectations of various variable instances, computed from the data given in [2] are listed in Table 1.

3. The Multi-State Inclusion-Exclusion Principle

The Inclusion-Exclusion (IE) Principle computes the cardinality of the union of n sets, through over-generous inclusion, followed by compensating exclusion. This principle remains valid when set cardinalities are replaced by finite probabilities. In reliability context, it is used for computing the probability of the union of n events, or equivalently the expectation of the disjunction (ORing) of the n indicator variables of such events. Usually, these indicator variables are products of instances of the underlying variables, which stand for the prime implicants P_i (called minimal paths) of system success, and the expectation of this success is the reliability of the system. With this interpretation, an application of the IE principle results in the following expression of reliability [10, 11]

$$R = E\{\vee_{i=1}^{n_p} P_i\} = \sum_{i=1}^{n_p} E\{P_i\} - \sum\sum_{1\leq i<j\leq n_p} E\{P_i \wedge P_j\} + \sum\sum\sum_{1\leq i<j<k\leq n_p} E\{P_i \wedge P_j \wedge P_k\} - \dots + (-1)^{n_p-1} E\{\wedge_{i=1}^{n_p} P_i\}.. \tag{9}$$

The number of terms in (8) is

$$\binom{n_p}{1} + \binom{n_p}{2} + \binom{n_p}{3} + \cdots + \binom{n_p}{n_p} = 2^{n_p} - 1, \quad (10)$$

i.e., it is exponential in the number of minimal paths. To apply the IE principle to (8) which has $n_p = 8$, we need 255 terms.

The IE principle is valid and applicable whether the implicants P_i and their constituting variables are two-valued or multi-valued. However, the implementation of (9) in the multi-state case needs to be aided by simplification rules for various products of the underlying variables. The IE simplicity is manifested in the fact that the simplification rule it requires (when handling coherent success) is just the following domination rule (which generalizes the idempotency rule of AND for an uncomplemented literal ($X_k \wedge X_k = X_k$) in the two-valued case)

$$X_k(\geq j_1)\, X_k(\geq j_2) = X_k(\geq j_2) \text{ for } j_2 \geq j_1, \quad (11a)$$

A similar simplification required by IE (when handling coherent failure) is the following domination rule (which is another generalization of the idempotency rule of AND for a complemented literal ($\bar{X}_k \wedge \bar{X}_k = \bar{X}_k$) in the two-valued case)

$$X_k(\leq j_1)\, X_k(\leq j_2) = X_k(\leq j_2) \quad \text{for } j_2 \leq j_1, \quad (11b)$$

Despite the great importance of the IE principle in combinatorics and probability theory, and despite its genuine conceptual simplicity, it does not seem to be the method of choice for evaluation of system reliability. It produces an exponential number of terms that have to be reduced subsequently via addition and cancellation. Moreover, it involves so many subtractions that make it highly sensitive to round-off errors in the ultra-reliable regime [10, 12–15]. For the problem of the running example, the symbolic computations are tedious, indeed. To show the reader a glimpse of how cumbersome this computation is, we show below the derivation of two (out of 255) of the terms involved, where repeated use is made of the domination rule (11a)

$$P_1 P_4 = (X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\})\, (X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{\geq 2\})$$

$$= X_2\{\geq 2\}\, X_3\{\geq 3\}\, X_4\{\geq 2\} X_5\{\geq 3\} X_7\{\geq 3\}\, X_8\{\geq 3\}, \quad (12a)$$

$$P_1 P_4 P_7 = (X_2\{\geq 2\}\, X_3\{\geq 3\}\, X_4\{\geq 2\} X_5\{\geq 3\} X_7\{\geq 3\}\, X_8\{\geq 3\})(X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_4\{\geq 2\} X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\})$$

$$= X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 3\}\, X_4\{\geq 2\} X_5\{\geq 3\} X_6\{\geq 2\} X_7\{\geq 3\}\, X_8\{\geq 3\}. \quad (12b)$$

The fact that the IE symbolic computations for the running example were terribly lengthy, made it highly error-prone. To make these computations perfect, we sought the guidance of a computer program written for the same purpose.

4. Multistate Probability-Ready Expressions

The concept of a probability-ready expression (PRE) is well-known in the two-valued logical domain [16], and it is still applicable for the multi-valued logical domain [17]. A Probability-Ready Expression is a random expression that can be directly transformed, on a one-to-one basis, to its statistical expectation (its probability of being equal to 1) by replacing all logic variables by their statistical expectations, and also replacing logical multiplication and addition (ANDing and ORing) by their arithmetic counterparts. A logic expression is a PRE if

a) all *ORed* products (terms formed by ANDing of literals) are *disjoint* (*mutually exclusive*),

b) all *ANDed* sums (alterms formed via ORing of literals) are *statistically independent*.

Condition (a) is satisfied if for every pair of ORed terms, there is at least a single opposition, *i.e.*, there is at least one variable that appears with a certain set of instances in one term and appears with a complementary set of instances in the other. Condition (b) is satisfied if for every pair of ANDed alterms (sums of disjunctions of literals), one alterm involves variables describing a certain set of components, while the other alterm depends on variables describing a set of different components (under the assumption of independence of components).

While there are many methods to introduce characteristic (a) of orthogonality (disjointness) into a multi-valued logic expression [1–3,18–21], there is no way to induce characteristic (b) of statistical independence. The best that one can do is to observe statistical independence when it exists, and then be careful not to destroy or spoil it and take advantage of it. Since one has the freedom of handling a problem from a success or a failure perspective, a choice should be made as to which of the two perspectives can more readily produce a PRE form. It is better to look at success for a system of no or poor redundancy (a series or almost-series system), and to view failure for a system of full or significant redundancy (a parallel or almost- parallel system) [10, 22, 23] .

The introduction of orthogonality might be achieved as follows. If neither of the two terms A and B in the sum $(A \vee B)$ subsumes the other ($A \vee B \neq A$ and $A \vee B \neq B$) and the two terms are not disjoint ($A \wedge B \neq 0$), then B can be disjointed with A by factoring out any common factor (using Boolean quotients) and then applying the *Reflection Law*, namely

$$\begin{aligned} A \vee B &= C((A/C) \vee (B/C)) \\ &= C((A/C) \vee \overline{(A/C)}(B/C)) \\ &= A \vee \overline{(A/C)}B. \end{aligned} \tag{13}$$

In (13), the symbol C denotes the common factor of $A\ and\ B$, and the Boolean quotient (A/C) might be viewed as the term A with its part common with B removed. If B subsumes A, then $C = A$ and $A/C = 1$, so that $\overline{(A/C)} = 0$, which means that B is absorbed in A. Note that (13) leaves the term A intact and replaces the term B by an expression that is disjoint with A. The quotient (A/C) is a product of e entities Y_k $(1 \leq k \leq e)$, so that $\overline{(A/C)}$ might be expressed via *De Morgan's Law* as a disjunction of the form

$$\overline{(A/C)} = \bigvee_{k=1}^{e} \bar{Y}_k. \tag{14}$$

Note that each Y_K is a literal that appears in the product A and does not appear in the product B. It stands for a disjunction of certain instances of some variable $X_{i(k)}\ and\ hence\ \bar{Y}_k$ is a disjunction of the complementary instances of the same variable. If we combine (13) with (14), we realize that the term B is replaced by e terms $(e \geq 1)$, which are each disjoint with the term A, but are not necessarily disjoint among themselves. Therefore, we replace the *De Morgan's Law* in (14) by a disjoint version of it [8], namely

$$\begin{aligned} &\overline{(A/C)} \\ &= \bar{Y}_1 \vee Y_1\bar{Y}_2 \vee Y_1Y_2\bar{Y}_3 \vee \ldots \vee Y_1Y_2 \ldots Y_{e-1}\bar{Y}_e \\ &= \bar{Y}_1 \\ &\vee Y_1\big(\bar{Y}_2 \\ &\vee Y_2(\bar{Y}_3 \vee \ldots \ldots \\ &\vee (\bar{Y}_{e-1} \\ &\vee Y_{e-1}\bar{Y}_e) \ldots)\big). \end{aligned} \tag{14a}$$

When (14a) is combined with (13), one obtains

$$\begin{aligned} A \vee = A \vee (&\bar{Y}_1 \vee Y_1\bar{Y}_2 \vee Y_1Y_2\bar{Y}_3 \vee \ldots \\ &\vee Y_1Y_2 \ldots Y_{e-1}\bar{Y}_e)B, \end{aligned} \tag{15}$$

where the first term A still remains intact, while the second term B is replaced by e terms which are each disjoint with A and are also disjoint among themselves. This means that one has a choice of either disjointing B with A in $A \vee B$, or disjointing A with B in $B \vee A$. The usual practice that is likely to yield good results is to order the terms in a given disjunction so that those with fewer literals should appear earlier.

The PRE concept is valid and applicable whether the products A and B as well as their constituting variables are two-valued or multi-valued. However, the implementation of (15) in the multi-state case needs to be aided by a few simplification rules for various products of the underlying variables. These simplification rules include the afore-mentioned two domination rules (11), the two differencing rules

$$X_k(\geq j_1)\, X_k(\leq j_2) = X_k(j_1, j_1+1, \dots, j_2) \quad \text{for } j_2 \geq j_1, \tag{16a}$$

$$X_k(\geq j_1)\, X_k(< j_2) = X_k(j_1, j_1+1, \dots, j_2-1) \quad \text{for } j_2 > j_1, \tag{16b}$$

which have no counterpart in the two-valued case, unless they are replaced by the orthogonality rules (which generalize the orthogonality ($X_k \wedge \bar{X}_k = 0$) in the two-valued case)

$$X_k(\geq j_1)\, X_k(\leq j_2) = 0 \text{ for } j_2 < j_1, \tag{16c}$$

$$X_k(\geq j_1)\, X_k(< j_2) = 0 \text{ for } j_2 \leq j_1, \tag{16d}$$

$$X_k(j)\, X_k(\neq j) = 0, \tag{16e}$$

and the complementation rules

$$\bar{X}_k\{\geq j\} = X_k\{< j\}, \tag{16f}$$

$$\bar{X}_k\{> j\} = X_k\{\leq j\}, \tag{16g}$$

$$\bar{X}_k\{j\} = X_k\{\neq j\}, \tag{16h}$$

Table 2 compares our initial success expression (8, rearranged) and its final PRE form, obtained after disjointing every original product with all succeeding products [24, 25]. The 8 products in (8, rearranged) have been replaced by 1+1+1+1+1+2+3+6 = 16 products. In a sense, the success expression remained 'shellable' up to its fifth term, while the sixth term was split into two terms, and the last two terms were replaced by three and six terms, respectively. The final multiplying factors introduced gradually via (12) and adjusted *via* (16) are distinguished in bold red in the right column of Table 2. What remains in black in this column is the variable instances that remained intact within an initial product**.**

Table 2. Comparison of the initial success expression (8a) in a minimal sum-of-product form with the final success expression in a probability-ready form

Initial success expression (Minimal s-o-p form)	Final success expression (PRE)
$X_3\{\geq 3\}\, X_7\{\geq 3\}$	$X_3\{\geq 3\}\, X_7\{\geq 3\}$
$\vee\ X_2\{\geq 3\}\, X_7\{\geq 3\}$	$\vee\ X_2\{\geq 3\}\, \boldsymbol{X_3\{< 3\}}\, X_7\{\geq 3\}$
$\vee\, X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}$	$\vee\, X_3\{\geq 3\}\, X_5\{\geq 3\}\, \boldsymbol{X_7\{< 3\}}\, X_8\{\geq 3\}$
$\vee\ X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}$	$\vee\ X_2\{\geq 3\}\, \boldsymbol{X_3\{< 3\}}\, X_5\{\geq 3\}$ $\boldsymbol{X_7\{< 3\}}\, X_8\{\geq 3\}$
$\vee\ X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_7\{\geq 3\}\, X_8\{\geq 2\}$	$\vee\ \boldsymbol{X_2\{2\}}\ \boldsymbol{X_3\{2\}}\, X_4\{\geq 2\}\, X_7\{\geq 3\}$ $X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 3\}$	$\vee\ X_1\{\geq 2\}\, \boldsymbol{X_2\{2\}\, X_3\{2\} X_4\{< 2\}}$ $X_6\{\geq 2\}\, X_7\{\geq 3\}$ $\vee\ X_1\{\geq 2\}\, \boldsymbol{X_2\{2\}\, X_3\{2\} X_4\{\geq 2\}}\, X_6\{\geq 2\}$ $X_7\{\geq 3\}\, \boldsymbol{X_8\{< 2\}}$
$\vee\ X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}$ $X_7\{\geq 2\}\, X_8\{\geq 2\}$	$\vee\ X_1\{\geq 2\}\boldsymbol{X_2\{< 2\}\, X_3\{2\}}\ X_4\{\geq 2\}$ $X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}$ $\vee\ X_1\{\geq 2\}\boldsymbol{X_2\{2\}\, X_3\{2\}}\ X_4\{\geq 2\}$ $X_6\{\geq 2\}\, \boldsymbol{X_7\{2\}}\, X_8\{\geq 2\}$ $\vee\ X_1\{\geq 2\}\boldsymbol{X_2\{\geq 3\}\, X_3\{2\}}\, X_4\{\geq 2\}$

	$X_5\{<3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$
$\vee\ X_1\{\geq 2\} X_2\{\geq 2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$	$\vee\ X_1\{\geq 2\} X_2\{\geq 3\} X_3\{2\} X_4\{\geq 2\} X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$ $\vee\ X_1\{\geq 2\} X_3\{\geq 3\} X_4\{\geq 2\} X_5\{<3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$ $\vee\ X_1\{\geq 2\} X_3\{\geq 3\} X_4\{\geq 2\} X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$ $\vee\ X_1\{\geq 2\} X_2\{2\} X_3\{<2\} X_4\{\geq 2\} X_6\{\geq 2\} X_7\{\geq 2\} X_8\{\geq 2\}$ $\vee\ X_1\{\geq 2\} X_2\{\geq 3\} X_3\{<2\} X_4\{\geq 2\} X_5\{<3\} X_6\{\geq 2\} X_7\{2\} X_8\{\geq 2\}$ $\vee\ X_1\{\geq 2\} X_2\{\geq 3\} X_3\{<2\} X_4\{\geq 2\} X_5\{\geq 3\} X_6\{\geq 2\} X_7\{2\} X_8\{2\}$.

5. The Multi-State Boole-Shannon Expansion

A prominent way for converting a Boolean formula into a PRE form is the Boole-Shannon Expansion, which takes the following form in the two-valued case [26].

$$f(\boldsymbol{X}) = (\bar{X}_i \wedge f(\boldsymbol{X}|0_i)) \vee (X_i \wedge f(\boldsymbol{X}|1_i)), \quad (17)$$

This Boole-Shannon Expansion expresses a (two-valued) Boolean function $f(\boldsymbol{X})$ in terms of its two sub-functions $f(\boldsymbol{X}|0_i)$ and $f(\boldsymbol{X}|1_i)$ These subfunctions are equal to the Boolean quotients $f(\boldsymbol{X})/\bar{X}_i$ and $f(\boldsymbol{X})/X_i$, and hence are obtained by restricting X_i in the expression of $f(\boldsymbol{X})$ to 0 and 1, respectively. If $f(\boldsymbol{X})$ is a function of n variables, the two sub-functions $f(\boldsymbol{X}|0_i)$ and $f(\boldsymbol{X}|1_i)$ are functions of at most $(n-1)$ variables. A possible (non-unique) multi-valued extension of (17) is [17]

$$S(\boldsymbol{X}) = X_i\{0\} \wedge (S(\boldsymbol{X})/X_i\{0\}) \vee X_i\{1\} \wedge (S(\boldsymbol{X})/X_i\{1\}) \vee X_i\{2\} \wedge (S(\boldsymbol{X})/X_i\{2\}) \vee X_i\{3\} \wedge (S(\boldsymbol{X})/X_i\{3\}) \vee \ldots \vee X_i\{m_i\} \wedge (S(\boldsymbol{X})/X_i\{m_i\}). \quad (18)$$

The expansion (18) serves our purposes very well. Once the sub-functions in (18) are expressed by PRE expressions, $S(\mathbf{X})$ will also be in PRE form, due to the combination of the following two facts:

(a) The R.H.S. of (18) has (m_i+1) disjoint terms, each of which containing one of the (m_i+1) disjoint instances $X_i\{0\}, X_i\{1\}, X_i\{2\}$, $X_i\{3\}$, …, and $X_i\{m_i\}$ of the variable X_i,

(b) Each of these (m_i+1) terms is a product of two statistically-independent entities, since any sub-function $S(\boldsymbol{X})/X_i\{j\}$ $(0 \leq j \leq m_i)$ does not involve any instance of the (m_i+1)-valued variable X_i, since its $X_i\{j\}$ instance is set to 1, while all its other instances are set to 0.

The expansion (18) might be viewed as a justification of the construction of the multi-valued Karnaugh map [27, 28]. This expansion transforms directly, on a one-to-one basis, to the probability domain as

$$E\{S(\boldsymbol{X})\} = E\{X_i\{0\}\} * E\{S(\boldsymbol{X})/X_i\{0\}\} + E\{X_i\{1\}\} * E\{S(\boldsymbol{X})/X_i\{1\}\} + E\{X_i\{2\}\} * E\{S(\boldsymbol{X})/X_i\{2\}\} + E\{X_i\{3\}\} * E\{S(\boldsymbol{X})/X_i\{3\}\} + \cdots + E\{X_i\{m_i\}\} * E\{S(\boldsymbol{X})/X_i\{m_i\}\}. \quad (19)$$

Equation (19) might be viewed as a restatement of the *Total Probability Theorem*, provided we interpret the expectation of a Boolean quotient as a conditional probability. It is the basis of multi-valued decision diagrams (*MDDs)*, that are optimally employed for the reliability analysis of multi-state systems, and that constitute the multi-valued counterpart of the Binary decision diagrams.

The expansion (18) is based on the orthonormal expansion set $\{X_i\{0\}, X_i\{1\}, \ldots, X_i\{m_i\}\}$, a set of disjoint and exhaustive elements. Any other orthonormal set (one of disjoint and exhaustive elements) might serve as an expansion basis for a different version of the multi-valued Boole-Shannon expansion other than (20). In the sequel, we will frequently use an orthonormal basis of the form $\{X_i\{< k\}, X_i\{k\}, X_i\{> k\}\}$.

We now apply variants of the expansion (18) to our running example. Employing, the reduced orthonormal set of expansion $\{X_7\{< 2\}, X_7\{2\}, X_7\{\geq 3\}\}$, we obtain the following Boole-Shannon expansion of S as given in (8a)

$$S = X_7\{< 2\}\ (S/\ X_7\{< 2\})\ \vee\ X_7\{2\}\ (S/\ X_7\{2\})\ \vee\ X_7\{\geq 3\}\ (S/\ X_7\{\geq 3\}). \quad (20)$$

Utilizing the relations

$$X_7\{\geq 3\}/X_7\{< 2\} = 0, \quad (21)$$

$$X_7\{\geq 2\}/X_7\{< 2\} = 0, \quad (22)$$

which result from orthogonality of $X_7\{< 2\}$ to each of $X_7\{\geq 3\}$ and $X_7\{\geq 2\}$, we apply the restriction $\{X_7\{< 2\} = 1\}$ to (8a). This replace (8a) by the following expression for $S/\ X_7\{< 2\}$

$$S/\ X_7\{< 2\} = X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}\ \vee\ X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}, \quad (23)$$

which is not a PRE, and hence we decompose it further using the orthonormal expansion set $\{X_3\{< 3\},\ X_3\{\geq 3\}\}$, namely

$$S/\ X_7\{< 2\} = X_3\{< 3\}\ (S/\ X_7\{< 2\}\, X_3\{< 3\})\ \vee\ X_3\{\geq 3\}\ (S/\ X_7\{< 2\}\, X_3\{\geq 3\}). \quad (24)$$

Utilizing the relations

$$X_3\{\geq 3\}/X_3\{< 3\} = 0, \quad (25)$$

$$X_3\{\geq 3\}/X_3\{\geq 3\} = 1, \quad (26)$$

we reduce (23) to the following expressions for $S/\ X_7\{< 2\}\, X_3\{< 3\}$ and $S/\ X_7\{< 2\}\, X_3\{\geq 3\}$

$$S/\ X_7\{< 2\}\, X_3\{< 3\} = X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}, \quad (27)$$

$$S/\ X_7\{< 2\}\, X_3\{\geq 3\} = X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} = X_5\{\geq 3\}\, X_8\{\geq 3\}, \quad (28)$$

where (28) is simplified through the absorption of the subsuming term $X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}$ in the subsumed term $X_5\{\geq 3\}\, X_8\{\geq 3\}$ (Recall that the set of literals in a subsuming term is a superset of the set of literals in a subsumed term). Subsequently, we rewrite (24) as

$$S/\ X_7\{< 2\} = \mathbf{X_3\{< 3\}}\ (X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}) \vee \mathbf{X_3\{\geq 3\}}\ (X_5\{\geq 3\} X_8\{\geq 3\}). \quad (29)$$

In retrospect, we note that the application of the disjointing operation (15) to (23) produces the following expression, which is simply a rearrangement of (29)

$$S/\ X_7\{< 2\} = (X_3\{\geq 3\} \vee X_2\{\geq 3\}\, \mathbf{X_3\{< 3\}})\, X_5\{\geq 3\}\, X_8\{\geq 3\}. \quad (30)$$

Now, we observe that the relations

$$X_7\{\geq 3\}/X_7\{2\} = 0, \quad (31)$$

$$X_7\{\geq 2\}/X_7\{2\} = 1, \quad (32)$$

result from the orthogonality of $X_7\{2\}$ to $X_7\{\geq 3\}$ and the fact that when we apply the restriction $\{X_7\{2\} = 1\}$ to $X_7\{\geq 2\} = X_7\{2\} \vee X_7\{\geq 1\}$ then $\{X_7\{\geq 2\} = 1\}$. Equations (31) and (32) lead to the replacement of (8a) by the following expression for $S/\ X_7\{2\}$

$$S/\ X_7\{2\} = X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\} X_2\{\geq 2\} X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}. \quad (33)$$

which is not a PRE, and hence we decompose it further using the orthonormal expansion set $\{X_8\{<2\}, X_8\{2\}, X_8\{\geq 3\}\}$, namely

$$S/X_7\{2\} = X_8\{<2\}\ (S/X_7\{2\}X_8\{<2\}) \vee X_8\{2\}(S/X_7\{2\}X_8\{2\}) \vee X_8\{\geq 3\}\ (S/X_7\{2\}X_8\{\geq 3\}). \quad (34)$$

Now, we utilize the relations

$$X_8\{\geq 3\}/X_8\{<2\} = 0, \quad (35)$$

$$X_8\{\geq 2\}/X_8\{<2\} = 0, \quad (36)$$

to apply the restriction $\{X_8\{<2\} = 1\}$ to (33), so as to discover that

$$S/X_7\{2\}X_8\{<2\} = 0. \quad (37)$$

Next, we use the relations

$$X_8\{\geq 3\}/X_8\{2\} = 0, \quad (38)$$

$$X_8\{\geq 2\}/X_8\{2\} = 1, \quad (39)$$

to apply the restriction $\{X_8\{2\} = 1\}$ to (33), and hence reduce it to the following expression for $S/X_7\{2\}X_8\{2\}$

$$S/X_7\{2\}X_8\{2\} = X_1\{\geq 2\}X_3\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\} \vee X_1\{\geq 2\}X_2\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\}, \quad (40)$$

which is not a PRE, but can be converted to such an expression as

$$S/X_7\{2\}X_8\{2\} = X_1\{\geq 2\}(X_2\{\geq 2\} \vee \boldsymbol{X_2\{<2\}}\ X_3\{\geq 2\})\ X_4\{\geq 2\}X_6\{\geq 2\}. \quad (41)$$

Now, we utilize the relations

$$X_8\{\geq 3\}/X_8\{\geq 3\} = 1, \quad (42)$$

$$X_8\{\geq 2\}/X_8\{\geq 3\} = 1, \quad (43)$$

to apply the restriction $\{X_8\{\geq 3\} = 1\}$ to (33), and hence reduce it to the following expression for $S/X_7\{2\}X_8\{\geq 3\}$

$$S_a = S/X_7\{2\}X_8\{\geq 3\} = X_3\{\geq 3\}X_5\{\geq 3\} \vee X_2\{\geq 3\}X_5\{\geq 3\} \vee X_1\{\geq 2\}X_3\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\} \vee X_1\{\geq 2\}X_2\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\}. \quad (44)$$

which is not a PRE, and hence we decompose it further using the following orthonormal expansion set that involves two variables: $\{X_2\{\geq 3\}, X_2\{<3\}X_3\{\geq 3\}, X_2\{<3\}X_3\{<3\}\}$. The decomposition involves the three subfunctions

$$S_a/\{X_2\{\geq 3\} = X_3\{\geq 3\}X_5\{\geq 3\} \vee X_5\{\geq 3\} \vee X_1\{\geq 2\}X_3\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\} \vee X_1\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\} = X_5\{\geq 3\} \vee X_1\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\}, \quad (45)$$

$$S_a/X_2\{<3\}X_3\{\geq 3\} = X_5\{\geq 3\} \vee 0 \vee X_1\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\} \vee X_1\{\geq 2\}\ (X_2\{2\}/X_2\{<3\})X_4\{\geq 2\}X_6\{\geq 2\}\} = X_5\{\geq 3\} \vee X_1\{\geq 2\}X_4\{\geq 2\}X_6\{\geq 2\}, \quad (46)$$

$$S_a/X_2\{<3\}X_3\{<3\} = X_1\{\geq 2\}\ (X_3\{2\}/X_3\{<3\})\ X_4\{\geq 2\}X_6\{\geq 2\} \vee X_1\{\geq 2\}\ (X_2\{2\}/X_2\{<3\})\ X_4\{\geq 2\}X_6\{\geq 2\} = X_1\{\geq 2\}((X_2\{2\}/X_2\{<3\}) \vee (X_3\{2\}/X_3\{<3\}))\ X_4\{\geq 2\}X_6\{\geq 2\}. \quad (47)$$

and hence (44) can be reduced to the PRE form (taking into consideration that $X_i\{<3\}(X_i\{2\}/X_i\{<3\} = X_i\{2\}$ since $X_i\{2\}(X_i\{2\}/X_i\{\geq 3\} = 0)$ $(X_2\{2\}/X_2\{<3\})$

$$S/X_7\{2\}X_8\{\geq 3\} = (X_2\{\geq 3\} \vee X_2\{<3\}X_3\{\geq 3\})(X_5\{\geq 3\} \vee X_1\{\geq 2\}X_4\{\geq 2\}\ \boldsymbol{X_5\{<3\}}X_6\{\geq 2\}) \vee X_1\{\geq 2\}(X_2\{2\}X_3\{<3\} \vee \boldsymbol{X_2\{<2\}}X_3\{2\})\ X_4\{\geq 2\}X_6\{\geq 2\}. \quad (48)$$

Combining (34), (37), (41) and (48), we obtain the following PRE for $S/X_7\{2\}$

$$S/X_7\{2\} = X_8\{2\}\ (X_1\{\geq 2\}(X_2\{\geq 2\} \vee X_2\{<2\}X_3\{\geq 2\})\ X_4\{\geq 2\}X_6\{\geq 2\})$$
$$\vee\ X_8\{\geq 3\}\ ((X_2\{\geq 3\} \vee X_2\{<3\}X_3\{\geq 3\})(X_5\{\geq 3\} \vee X_1\{\geq 2\}X_4\{\geq 2\}X_5\{<3\}X_6\{\geq 2\}) \vee X_1\{\geq 2\}(X_2\{2\}X_3\{<3\} \vee X_2\{<2\}X_3\{2\})\ X_4\{\geq 2\}X_6\{\geq 2\}). \quad (49)$$

Now, we express $S_b = S/X_7\{\geq 3\}$ as

$$S_b = S/X_7\{\geq 3\} = X_3\{\geq 3\} \vee X_2\{\geq 3\}$$

$\vee X_3\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\}\, X_2\{\geq 3\}\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_6\{\geq 2\} \vee X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_4\{\geq 2\} X_6\{\geq 2\}\, X_8\{\geq 2\}$

$$= X_3\{\geq 3\} \vee X_2\{\geq 3\} \vee X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_6\{\geq 2\} \vee X_1\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_2\{\geq 2\} X_4\{\geq 2\} X_6\{\geq 2\}\, X_8\{\geq 2\}. \quad (50)$$

which is not a PRE, and hence we decompose it further using the following orthonormal expansion set that involves two variables: $\{X_2\{\geq 3\}, X_2\{< 3\}X_3\{\geq 3\}, X_2\{< 3\}X_3\{< 3\}\}$. The decomposition involves the three subfunctions

$$S_b/\{X_2\{\geq 3\} = 1, \quad (51)$$

$$S_b/X_2\{< 3\}X_3\{\geq 3\} = 1, \quad (52)$$

$$S_b/X_2\{< 3\}X_3\{< 3\} = (X_2\{2\}/X_2\{< 3\})\, (X_3\{2\}/X_3\{< 3\})\, X_4\{\geq 2\}\, X_8\{\geq 2\}$$

$$\vee\ X_1\{\geq 2\}\, (X_2\{2\}/X_2\{< 3\})(X_3\{2\} /X_3\{< 3\})\, X_6\{\geq 2\}$$

$$\vee\ X_1\{\geq 2\}\, (X_3\{2\} /X_3\{< 3\})\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}$$

$$\vee\ X_1\{\geq 2\}\, (X_2\{2\} /X_2\{< 3\})\, X_4\{\geq 2\} X_6\{\geq 2\}\, X_8\{\geq 2\}$$

$$= (X_2\{2\}/X_2\{< 3\})(X_3\{2\}/X_3\{< 3\})\, (X_4\{\geq 2\}\, X_8\{\geq 2\} \vee X_1\{\geq 2\}\, X_6\{\geq 2\}) \vee X_1\{\geq 2\}\, ((X_2\{2\}/X_2\{< 3\}) \vee (X_3\{2\}/X_3\{< 3\}))\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}. \quad (53)$$

Now, we rewrite (53) in the PRE form

$$X_2\{< 3\}X_3\{< 3\}\, (S_b/X_2\{< 3\}X_3\{< 3\}) =$$

$$X_2\{2\}\, X_3\{2\}\, (X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (\boldsymbol{X_4}\{< 2\} \vee \boldsymbol{X_4}\{\geq 2\}\, \boldsymbol{X_8}\{< 2\})\, X_1\{\geq 2\}\, X_6\{\geq 2\}) \vee X_1\{\geq 2\}\, (X_2\{2\}\boldsymbol{X_3}\{< 2\} \vee \boldsymbol{X_2}\{< 2\}X_3\{2\})\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}. \quad (54)$$

Utilizing (51), (52) and (54), we obtain $S/\, X_7\, \{\geq 3\}$ in PRE form as

$$S/\, X_7\, \{\geq 3\} = X_2\{\geq 3\} \vee X_2\{< 3\}\, X_3\{\geq 3\} \vee X_2\{2\}\, X_3\{2\}\, (X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (X_4\{< 2\} \vee X_4\{\geq 2\}\, X_8\{< 2\})\, X_1\{\geq 2\}\, X_6\{\geq 2\})$$

$$\vee\ X_1\{\geq 2\}\, (X_2\{2\}X_3\{< 2\} \vee X_2\{< 2\}X_3\{2\})\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}. \quad (55)$$

The final required PRE expression is obtained via (20), (30), (49) and (55)

$$S = X_7\, \{< 2\}\, (X_3\{\geq 3\} \vee X_2\{\geq 3\}\, X_3\{< 3\}\,)\, X_5\{\geq 3\}\, X_8\{\geq 3\} \vee$$

$$X_7\, \{2\}\, (X_8\, \{2\}\, (X_1\{\geq 2\}(X_2\{\geq 2\} \vee X_2\{< 2\}\, X_3\{\geq 2\})\, X_4\{\geq 2\}\, X_6\{\geq 2\})$$

$$\vee\ X_8\, \{\geq 3\}\, ((X_2\{\geq 3\} \vee X_2\{< 3\}X_3\{\geq 3\})\, (X_5\{\geq 3\} \vee X_1\{\geq 2\}\, X_4\{\geq 2\}\, X_5\{< 3\}X_6\{\geq 2\}) \vee X_1\{\geq 2\}(X_2\{2\}X_3\{< 3\} \vee X_2\{< 2\}X_3\{2\})\, X_4\{\geq 2\}\, X_6\{\geq 2\}) \vee$$

$$X_7\, \{\geq 3\}\, (X_2\{\geq 3\} \vee X_2\{< 3\}\, X_3\{\geq 3\} \vee$$

$$X_2\{2\}\, X_3\{2\}\, (X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (X_4\{< 2\} \vee X_4\{\geq 2\}\, X_8\{< 2\})\, X_1\{\geq 2\}\, X_6\{\geq 2\})$$

$$\vee\ X_1\{\geq 2\}\, (X_2\{2\}X_3\{< 2\} \vee X_2\{< 2\}X_3\{2\})\, X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_8\{\geq 2\}). \quad (56)$$

This PRE is converted, on a one-to-one basis, into an expectation, by replacing each Boolean variable and Boolean operator by its arithmetic counterpart, namely

$$E\{S\} = E\{X_7\, \{< 2\}\}\, (E\{X_3\{\geq 3\}\} + E\{X_2\{\geq 3\}\}\, E\{X_3\{< 3\}\})\, E\{X_5\{\geq 3\}\}\, E\{X_8\{\geq 3\}\} +$$

$$E\{X_7\, \{2\}\}\, (E\{X_8\, \{2\}\}\, (E\{X_1\{\geq 2\}\}\, (E\{X_2\{\geq 2\}\} + E\{X_2\{< 2\}\}E\{X_3\{\geq 2\}\})\, E\{X_4\{\geq 2\}\}\, E\{X_6\{\geq 2\}\}) + E\{X_8\, \{\geq 3\}\}\, (\mathrm{E}\{(X_2\{\geq 3\}\} + E\{X_2\{< 3\}\}\, E\{X_3\{\geq 3\}\})\, (E\{X_5\{\geq 3\}\} + E\{X_1\{\geq 2\}\}\, E\{X_4\{\geq 2\}\}\, E\{X_5\{< 3\}\}\, E\{X_6\{\geq 2\}\}) + E\{X_1\{\geq 2\}\}(E\{X_2\{2\}\}\, E\{X_3\{< 3\}\} + E\{X_2\{< 2\}\}\, E\{X_3\{2\}\})\, E\{X_4\{\geq 2\}\}\, E\{X_6\{\geq 2\}\}) +$$

$E\{X_7\{\geq 3\}\}\,(E\{X_2\{\geq 3\}\} + E\{X_2\{< 3\}\}\,E\{X_3\{\geq 3\}\} +$

$E\{X_2\{2\}\}E\{X_3\{2\}\}\,(E\{X_4\{\geq 2\}\}\,E\{X_8\{\geq 2\}\} + (E\{X_4\{< 2\}\} + E\{X_4\{\geq 2\}\}\,E\{X_8\{< 2\}\})\,E\{X_1\{\geq 2\}\}\,E\{X_6\{\geq 2\}\}) + E\{X_1\{\geq 2\}\}(E\{X_2\{2\}\}E\{X_3\{< 2\}\} + E\{X_2\{< 2\}\}E\{X_3\{2\}\})\,E\{X_4\{\geq 2\}\}E\{X_6\{\geq 2\}\}E\{X_8\{\geq 2\}\}).$ (56a)

In retrospect, we note that our choice of the decomposition set $\{X_7\{< 2\}, X_7\{2\}, X_7\{\geq 3\}\}$ in (20) is warranted by the definite simplification achieved via the Boolean quotients in (21), (22), (25), (26), (31), and (32). Had we employed a smaller decomposition set $\{X_7\{< 2\},\ X_7\{\geq 2\}\}$, we would have encountered a Boolean quotient of the form $(X_7\{\geq 3\}/X_7\{\geq 2\})$, which has no simple form. Similarly, if we had adopted, instead, the two-element decomposition set $\{X_7\{< 3\}, X_7\{\geq 3\}\}$, we would have obtained a Boolean quotient of the form $(X_7\{\geq 2\}/X_7\{< 3\})$, which also does not possess a simple form.

6. Inclusion-Exclusion for Composite PRE Paths

The literature abounds with innovative attempts to mitigate the shortcomings of the multi-state inclusion-exclusion (MS-IE) procedure [11, 29–32]. This section offers yet another attempt along this direction. The success expression (8a) is rewritten in the factored form

$$S = \Re_1 \vee \Re_2 \vee \Re_3 = (X_2\{\geq 3\} \vee X_3\{\geq 3\})\,(X_7\{\geq 3\} \vee X_5\{\geq 3\}\,X_8\{\geq 3\})$$
$$\vee\ X_1\{\geq 2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}\,(X_2\{\geq 2\} \vee X_3\{\geq 2\})$$
$$\vee\ X_2\{\geq 2\}\,X_3\{\geq 2\}\,X_7\{\geq 3\}\,(X_4\{\geq 2\}\,X_8\{\geq 2\} \vee X_1\{\geq 2\}\,X_6\{\geq 2\}), \quad (57)$$

which comprises three rather than eight implicants or paths, and hence it has an IE formula of just 7 (rather than 255) terms, namely

$$E\{S\} = E\{\Re_1\} + E\{\Re_2\} + E\{\Re_3\} - E\{\Re_1\Re_2\} - E\{\Re_1\Re_3\} - E\{\Re_2\Re_3\} + E\{\Re_1\Re_2\Re_3\}. \quad (58)$$

However, this dramatic reduction in the number of terms comes at a price, namely, the implicant products in (58) are not necessarily in PRE form, and must be recast as such. Fortunately, the required cost is very modest indeed. The first implicant is a product of two statistically independent expressions, each of which is easily converted into a PRE, namely

$$\Re_1 = (X_2\{\geq 3\} \vee \mathbf{X_2\{< 3\}}X_3\{\geq 3\})\ \ (X_7\{\geq 3\} \vee X_5\{\geq 3\}\,\mathbf{X_7\{< 3\}}\,X_8\{\geq 3\}). \quad (59)$$

Likewise, the two other implicants are easily converted into PREs, *viz.*

$$\Re_2 = X_1\{\geq 2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}\,(X_2\{\geq 2\} \vee \mathbf{X_2\{< 2\}}X_3\{\geq 2\}). \quad (60)$$

$$\Re_3 = X_2\{\geq 2\}\,X_3\{\geq 2\}X_7\{\geq 3\}$$
$$(X_4\{\geq 2\}\,X_8\{\geq 2\} \vee (\mathbf{X_4\{< 2\} \vee X_4\{\geq 2\}X_8\{< 2\}})X_1\{\geq 2\}\,X_6\{\geq 2\}). \quad (61)$$

Products of these implicants inherit the PRE property without further processing. They just need simplification via the domination rules (11)

$$\Re_1\,\Re_2 = (X_2\{\geq 3\} \vee \mathbf{X_2\{< 3\}}X_3\{\geq 3\})\ \ (X_2\{\geq 2\} \vee \mathbf{X_2\{< 2\}}X_3\{\geq 2\})\,X_1\{\geq 2\}$$
$$X_4\{\geq 2\}\,X_6\{\geq 2\}\,X_7\{\geq 2\}\,X_8\{\geq 2\}\,(X_7\{\geq 3\} \vee \mathbf{X_7\{< 3\}}X_5\{\geq 3\}\,X_8\{\geq 3\})$$
$$= (X_2\{\geq 3\} \vee \mathbf{X_2\{< 3\}}X_3\{\geq 3\})\,X_1\{\geq 2\}\,X_4\{\geq 2\}\,X_6\{\geq 2\}$$
$$(X_7\{\geq 3\}\,X_8\{\geq 2\} \vee X_5\{\geq 3\}\ \mathbf{X_7\{2\}}\,X_8\{\geq 3\}). \quad (62)$$

$$\Re_1\,\Re_3 = (X_2\{\geq 3\} \vee \mathbf{X_2\{< 3\}}X_3\{\geq 3\})\,(X_7\{\geq 3\} \vee X_5\{\geq 3\}\,\mathbf{X_7\{< 3\}}\,X_8\{\geq 3\})\,X_2\{\geq 2\}$$
$$(X_4\{\geq 2\}\,X_8\{\geq 2\} \vee (\mathbf{X_4\{< 2\} \vee X_4\{\geq 2\}X_8\{< 2\}})\,X_1\{\geq 2\}\,X_6\{\geq 2\})\ \ X_3\{\geq 2\}X_7\{\geq$$

$3\} = (X_2\{\geq 3\}X_3\{\geq 2\} \vee \boldsymbol{X_2\{2\}}\, X_3\{\geq 3\})\ X_7\{\geq 3\}$

$(X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (\boldsymbol{X_4\{< 2\} \vee X_4\{\geq 2\}X_8\{< 2\}})\ X_1\{\geq 2\}\, X_6\{\geq 2\}).$ (63)

$\Re_2\, \Re_3 = X_1\{\geq 2\}\ X_4\{\geq 2\}\, X_6\{\geq 2\}\, X_7\{\geq 2\}\, X_8\{\geq 2\}\ (X_2\{\geq 2\} \vee \boldsymbol{X_2\{< 2\}}X_3\{\geq 2\})\ X_2\{\geq 2\}$

$X_3\{\geq 2\}\, X_7\{\geq 3\}\ (X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (\boldsymbol{X_4\{< 2\} \vee X_4\{\geq 2\}X_8\{< 2\}})\ X_1\{\geq 2\}\, X_6\{\geq 2\}) = X_1\{\geq 2\}X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}\ X_6\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}.$ (64)

$\Re_1\Re_2\Re_3 = (\Re_1\Re_2)(\Re_1\Re_3)(\Re_2\Re_3) = (X_2\{\geq 3\} \vee \boldsymbol{X_2\{< 3\}}X_3\{\geq 3\})\ X_1\{\geq 2\}X_4\{\geq 2\}\, X_6\{\geq 2\}\ (\, X_7\{\geq 3\}\, X_8\{\geq 2\} \vee X_5\{\geq 3\}\, \boldsymbol{X_7\{2\}}\, X_8\{\geq 3\})\ (X_2\{\geq 3\}X_3\{\geq 2\} \vee \boldsymbol{X_2\{2\}}\, X_3\{\geq 3\})\ X_7\{\geq 3\}$

$(X_4\{\geq 2\}\, X_8\{\geq 2\} \vee (\boldsymbol{X_4\{< 2\} \vee X_4\{\geq 2\}X_8\{< 2\}})\ X_1\{\geq 2\}\, X_6\{\geq 2\})$

$X_1\{\geq 2\}X_2\{\geq 2\}\, X_3\{\geq 2\}\, X_4\{\geq 2\}$

$X_6\{\geq 2\}X_7\{\geq 3\}\, X_8\{\geq 2\}$

$= (X_2\{\geq 3\}\boldsymbol{X_3\{\geq 2\}} \vee \boldsymbol{X_2\{2\}}X_3\{\geq 3\})\ X_1\{\geq 2\}\, X_4\{\geq 2\}\ X_6\{\geq 2\}X_7\{\geq 3\}\ X_8\{\geq 2\}.$ (65)

7. Discussion

In this paper, we presented four methods (in descending order of computational complexity) for solving the problem of our running example. Table 3 indicates that our four methods agree on a value of 0.9819022224313, which is a more precise version for the value of network reliability (0.981902) that was obtained earlier by Lin *et al.* [2]. Of course, the exaggerated precision of the values in Table 3 is not practically warranted, but it instils confidence in the correctness of the various computational methods.

Table 3. Comparison of numerical results obtained for the same running example

Method	Numerical Result
RSDP (Lin *et al*., 2014)	0.981902
Conventional IE (Equation (9))	0.98190222243132085
PRE (Table 2)	0.981902222431299
MS-BS expansion (Equation (56a))	0.981902222431299294
IE improved with PRE (Equation (58))	0.9819022224312988074

8. Conclusions

This paper is a continuation of our earlier efforts to extend the concept of the sum of disjoint products (SDP) in the domain of multi-state reliability to the more encompassing one of a probability-ready expression (PRE). The paper served as an exposition of the inter-relationships among the multi-state concepts MS-IE, MS-PRE, and MS-BS. This exposition was obtained by applying four related standard or novel approaches to the same problem of multi-state network reliability. Each of these approaches recovered the same result obtained by the RSDP method.

Conflict of Interest

The authors assert that no conflict of interest exists.

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الارتباط بين مبدأ الشمول والاستبعاد والتعبيرات الجاهزة للاحتمال ومفكوك بول–شانون لأجل المعولية متعددة الحالات

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المستخلص. تتناول هذه الورقة صورة مستجدة للمسألة التقليدية لحساب احتمال اتحاد ن من الأحداث، وهو ما يعادل توقع دالة الفصل لمتغيرات مؤشرات التبيين لهذه الأحداث، أي احتمال أن تكون دالة الفصل هذه مساويةً للواحد. تتعامل صورة هذه المسألة المدروسة هنا مع المتغيرات متعددة القيم، حيث يشير الاحتمال المطلوب إلى معولية شبكة توصيل متعددة الحالات (ش وع ح)، وهي شبكة يوصف نجاح نظامها بدالة ثنائية القيمة يتم التعبير عنها بدلالة النجاحات متعددة القيم لعناصر النظام. تناقش الورقة أربعة مناهج لدراسة المسألة المذكورة أعلاه بدلالة ش و ع ح قياسية، يُعرف نجاحها في صورة أصغرية بدالة الفصل لضامناتها الأولية، التي تمثل المسارات الأصغرية للشبكة ذات الصلة. تحدد الورقة بإيجاز وتناقش حلين قياسيين باستخدام مبدأ الشمول والاستبعاد متعدد الحالات (ش ب–ع ح) وبإنشاء تعبير جاهز للاحتمال متعدد الحالات (ع ج ح–ع ح). نجحنا في استقراء وتوسعة مفهوم ع ج ح–ع ح من المجال المنطقي ذي القيمتين إلى المجال المنطقي متعدد القيم، وفي استخدامه للتحويل المباشر للتعبير المنطقي العشوائي، على أساس واحد لواحد، إلى شكل توقع إحصائي، ببساطة من خلال تغيير جميع المتغيرات المنطقية إلى توقعاتها الإحصائية، وكذلك استبدال الضرب والجمع الحسابي بنظيريهما المنطقيين (دالتي العطف والفصل). تتمثل المساهمة الرئيسة لهذه الورقة في توفير إجراءين منظمين وأكثر كفاية للتعامل مع المسألة المطلوبة. يستخدم الإجراء الأول مفكوك بول–شانون متعدد الحالات، بينما يطبق الإجراء الثاني مبدأ ش ب–ع ح على عدد أقل من المسارات الأصغرية (المحللة أو المركبة) التي تم وضعها (بأقل تكلفة) على صيغة ع ج ح–ع ح. يتم توضيح الطرائق الأربع التي تمت مناقشتها بمثال رمزي مفصل لدراسة حالة حقيقية، وقد أنتج كل منها نسخة أكثر دقة لنفس القيمة العددية التي تم الحصول عليها مسبقًا بطريقة المجموع المعاود للمضروبات المتنافية (ج ع ض ن). تعد هذه الورقة جزءًا من نشاط مستمر يسعى إلى توفير معالجة تعليمية لمسائل المعولية متعددة الحالات، وإلى إنشاء علاقة متبادلة واضحة وثاقبة بين نمذجة الحالتين والنمذجة متعددة

الحالات من خلال التأكيد على أن المفاهيم عديدة القيمة هي امتدادات طبيعية وبسيطة للمفاهيم ذات القيمتين.

الكلمات المفتاحية: معولية الشبكة، الشمول والاستبعاد، التعبير الجاهز للاحتمالية، مفكوك بول– شانون، النظام متعدد الحالات، المنطق متعدد القيم، التعبير الرمزي، شبكة التسليم متعددة الحالات.