KING ABDULAZIZ UNIVERSITY

DEPARTMENT OF MATHEMATICS

PhD Entrance Exam

Second Semester 1441

Maximim Time: Three Hours

*Name:*____

Student number: ________________________________

Field of specialization: _____________________________

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8

Note : Please answer 4 questions of 8.

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Q1) Solve the linear system

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 &= 7 \\ 2x_1 - 4x_2 - x_3 &= 1 \\ -x_1 + x_2 + 4x_3 &= -5 \end{aligned}$$

using Gaussian elimination with back substitution.

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Q2) Find the steady states, determine their stability and classify their type (node, focus, saddle,...)

$$\dot{x} = 3x - x^2 - 2xy,$$
$$\dot{y} = 2y - xy - y^2.$$

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Q3) Let λ denote the Lebesgue measure on $\mathbb{R}$, λ^* denote the Lebesgue outer measure on $\mathbb{R}$. Let $\mathcal{M}$ denote the $\sigma-$ algebra of Lebesgue measurable subsets of $\mathbb{R}$.

(a) For $\epsilon > 0$. Prove there is an open set O such that $A \subseteq O$ with $\lambda(O) = \lambda^*(O) < \lambda^*(A) + \epsilon$.

(b) If A, B are subset of $\mathbb{R}$. Prove that $\lambda^*(A \cup B) + \lambda^*(A \cap B) \leqslant \lambda^*(A) + \lambda^*(B)$.

You may use the fact that if $A, B \in \mathcal{M}$. **Then** $\lambda(A \cup B) + \lambda(A \cap B) = \lambda(A) + \lambda(B)$.

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Q4) The topology $\mathbb{R}_l$ generated by basis elements $[a, b)$ is called Sorgenfrey topology on $\mathbb{R}$. Prove that it is a Lindelof space. But product topology Sorgenfrey plane generated by rectangles $[a, b)[c, d)$ is not Lindelof. A Lindelof space is a space whose every open cover has a countable subcover.

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Q5) (a) Let $V = R^3$ be a vector space over the field of real numbers R. Determine whether $W = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_1\lambda_2 = 0\}$ is a subspace of V or not. Justify your answer.
(b) Let $V = R^3$ be a vector space over the field of real numbers R and let $x_1 = (1,0,0)$, $x_2 = (1,1,0)$ and $x_3 = (1,1,1)$. Show that $V = R_{x_1} + R_{x_2} + R_{x_3}$. Moreover, show that $V = \oplus\sum_{i=1}^{3} R_{x_i}$.

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Q6) (a) Determine whether the existence and uniqueness theorem does or does not guarantee the existence of solutions for the following initial value problems. If the existence is ensured, then check the uniqueness of that solution:

$$(i) \quad \frac{dy}{dx} = \ln(1+y^2), \ y(0) = 0;$$

$$(ii) \quad y\frac{dy}{dx} = x - 1, \ y(1) = 0.$$

(b) Solve the system

$$x_1'(t) = x_1 - 3x_2, \ \ x_1(0) = 1,$$
$$x_2'(t) = 3x_1 + 7x_2, \ \ x_2(0) = 0.$$

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Q7) **(a)** Prove that on a finite dimensional vector space X, any norm $||.||$ is equivalent to any other norm $||.||_0$.

(b) Consider $C[0,1]$ with norm $||x|| = \max_{t\in[0,1]} |x(t)|$ and let $\mu > 0$. Define $f : C[0,1] \to \mathbb{R}$ by

$$f(x) = \mu \int_0^1 x(t)dt.$$

Show that f is a bounded linear functional with $||f|| = \mu$.

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Q8) (a) Give an example of a complex number which is not a real number.

(b) Find the real and imaginary parts of the complex number $(i+1)^{-1}$.

(c) State the de Moivre's Formula.